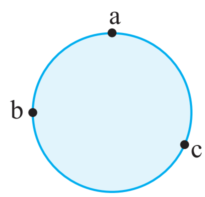
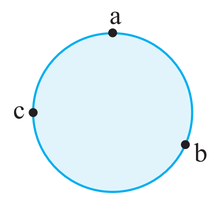
CIRCULAR PERMUTATION Section - 7

7.1 Introduction to Circular Permutation

When objects are to be arranged (ordered) in a circle instead of a row, it is known as Circular Permutation.

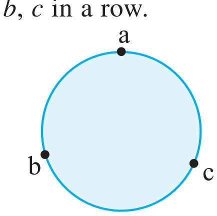
For example, three objects a, b, c can be permutated in a circle as shown below :



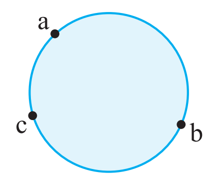


Number of ways to arrange a, b, c in circle is not same as number of ways to arrange a, b, c in a row.

This is because arrangements *abc, bca, cab* in a row are same in circle as shown in the figure on right.



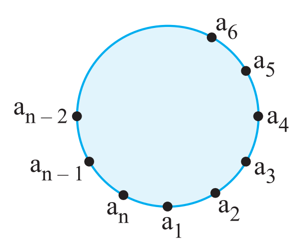
Similarly, arrangements *acb, cba, bac* in a row are same in circle as shown in the figure on right.



Therefore,

Number of ways to arrange n different objects in a circle = Number of circular permutations of n objects = $\lfloor n-1$.

Proof :



Let $a_1, a_2, a_3, \ldots\ldots a_{n-1}, a_n$, be n distinct objects. Let the total number of circular permutations be x. Consider one of these x permutations as shown in Figure.

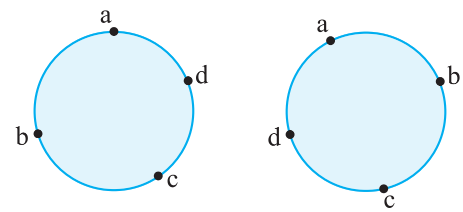
Clearly, this circular permutation provides n linear permutations as given below :

$$a_1, a_2, a_3, \ldots\ldots\ldots a_{n-1}, a_n$$
$$a_2, a_3, a_4, \ldots\ldots\ldots a_n, a_1$$
$$a_3, a_4, a_5, \ldots\ldots\ldots a_n, a_1, a_2$$
$$a_4, a_5, a_6, \ldots\ldots\ldots a_n, a_1, a_2, a_3$$
$$\cdots \qquad \cdots$$
$$\cdots \qquad \cdots$$
$$a_n, a_1, a_2, a_3, \ldots\ldots, a_{n-1}$$

Thus, each circular permutation gives n linear permutations. But there are x circular permutations. So, that number of linear permutatiosn is xn. But the number of linear permutations of n distinct objects is n !.

$$\therefore \qquad xn = n! \quad \Rightarrow \quad x = \frac{n!}{n} = (n-1)!$$

7.2 Difference between Clockwise and Anti Clock wise



In some of the problems, we need to consider clockwise and anti-clockwise arrangements of objects as same arrangements.

See the following circular permutations.

There is a difference of just the cyclic order. In first arrangement a, b, c, d are arranged in anti-clockwise order where as in second these are arranged in clockwise order.

If we have to consider these arrangements same (for example, arrangement of flowers in garland or arrangement of beads in a necklace), then we need to divide total circular permutations by 2.

Therefore,

Number of circular permutations of n different objects such that clockwise and anticlockwise arrangements of objects are same $= \dfrac{|\underline{n-1}}{2}$.

Illustrating the Concepts :

➤ *In how many ways can a party of* 4 *men and* 4 *women be seated at a circular table so that no two women are adjacent?*

(A) 144 **(B)** 24 **(C)** 72 **(D)** 288

The 4 men can be seated at the circular table such that there is a vacant seat between every pair of men in (4 – 1)! = 3! ways. Now, 4 vacant seats can be occupied by 4 women in 4! ways.

Hence, the required number of seating arrangements = 3! × 4! = 144

➤ *In how many ways can seven persons sit around a table so that all shall not have the same neighbours in any two arrangements* ?

(A) 360 **(B)** 180 **(C)** 720 **(D)** 90

Clearly, 7 persons can sit at a round table in (7 – 1)! = 6! ways. But, in clockwise and anticlockwise arrangements, each person will have the same neighbours.

So, the required number of ways $= \frac{1}{2}(6!) = 360$

Illustration - 47 *There are* 20 *persons among whom are two brothers. Find the number of ways in which we can arrange them around a circle so that there is exactly one person between the two brothers.*

(A) 18! **(B)** $2 \times 18!$ **(C)** 17! **(D)** $2 \times 17!$

SOLUTION : (B)

Let B_1 and B_2 be two brothers among 20 persons and let M be a person that will sit between B_1 and B_2. The person M can be chosen from 18 persons (excluding B_1 and B_2) in 18 ways. Considering the two brothers B_1 and B_2 and person M as one person, we have 18 persons in all. These 18 persons can be arranged around a circle in (18 – 1)! = 17! ways. But B_1 and B_2 can be arranged amongst themselves in 2! ways.

Hence, the total number of ways = 18 × 17! × 2! = 2 × 18!.

Illustration - 48 *A round table conference is to be held between 20 delegates of 2 countries. In how many ways can they be seated if two particular delegates are*

(i) *always together?*

(A) 18! **(B)** $2 \times 18!$ **(C)** 17! **(D)** $2 \times 17!$

(ii) *never together?*

(A) $17 \times 18!$ **(B)** $18 \times 18!$ **(C)** $15! \times 18!$ **(D)** $16! \times 18!$

SOLUTION : (i).(B) (ii).(A)

(i) Let D_1 and D_2 be two particular delegates. Considering D_1 and D_2 as one delegate, we have 19 delegates in all. These 19 delegates can be seated round a circular table in $(19 - 1)! = 18!$ ways. But two particular delegates can arrange amongst themselves in 2! ways ($D_1\ D_2$ and $D_2\ D_1$).

Hence, the total number of ways $= 18! \times 2! = 2\ (18!)$

(ii) To find the number of ways in which two particular delegates never sit together, we subtract the number of ways in which they sit together from the total number of seating arrangements of 20 persons around the round table. Clearly 20 persons can be seated around a circular table in $(20 - 1)! = 19!$ ways.

Hence, the required number of seating arrangements $= 19! - 2 \times 18! = 17\ (18!)$

Another Approach :

First arrange remaining 18 persons in $(18 - 1)\ ! = 17!$ ways.

Then select two places out of 18 places in ${}^{18}C_2$ ways and arrange the two in 2! ways.

No. of ways $= 17! \times {}^{18}C_2 \times 2! = 17\ (18!)$

Illustration - 49 *There are 5 gentlemen and 4 ladies to dine at a round table. In how many ways can they seat themselves so that no two ladies are together ?*

(A) 2880 **(B)** 1440 **(C)** 720 **(D)** 360

SOLUTION : (A)

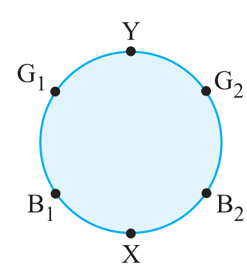
Five gentlemen can be seated at a round table in $(5 - 1)! = 4!$ ways. Now, 5 places are created in which 4 ladies are to be seated. Select 4 seats for 4 ladies from 5 seats in ${}^5C_4$ ways. Now 4 ladies can be arranged on the 4 selected seats in 4! ways. Hence, the total number of ways in which no two ladies sit together $= 4! \times {}^5C_4 \times 4! = (4!)\ 5\ (4!) = 2880$

Illustration - 50 *Three boys and three girls are to be seated around a table in a circle. Among them, the boy X does not want any girl neighbour and the girl Y does not want any boy neighbour. How many such arrangements are possible?*

(A) 4 **(B)** 8 **(C)** 2 **(D)** 16

SOLUTION : (A)

Let B_1, B_2 and X be three boys and G_1, G_2 and Y be three girls. Since the boy X does not want any girl neighbour. Therefore boy X will have his neighbours as boys B_1 and B_2 as shown in Fig. Similarly, girl Y has her neighbours as girls G_1 and G_2 as shown in the figure. But the boys B_1 and B_2 can be arranged amongst themselves in 2! ways and the girls G_1 and G_2 can be arranged amongst themselves in 2! ways.



Hence, the required number of arrangements $= 2! \times 2! = 4$

Illustration - 51 *Find the number of ways in which 8 different flowers can be strung to form a garland so that 4 particular flowers are never separated.*

(A) 288 **(B)** 144 **(C)** 72 **(D)** 576

SOLUTION : (A)

Considering 4 particular flowers as one group of flowers, we have five flowers (one group of flowers and remaining four flowers) which can be strung to form a garland in 4!/2 ways. But 4 particular flowers can be arranged amongst themselves in 4! ways. Thus, the required number of ways = $\frac{4! \times 4!}{2} = 288$

DIVISION OF IDENTICAL OBJECTS INTO GROUPS — Section - 8

8.1 Introduction

In this section, we will discuss how to find number of ways to divide identical objects into groups.

For example, if we have to divide 5 identical copies of a book among 3 boys such that each boy gets at least 1 copy, then it can be achieved as shown below :

Boy 1	Boy 2	Boy 3
3	1	1
1	3	1
1	1	3
2	2	1
1	2	2
2	1	2

i.e. 5 identical copies can be divided in 6 ways.

We can study following formulae to find number of ways to divide them instead of writing ways and counting them.

8.2 Formulae

(a) The number of ways to divide n identical objects into r groups (different) such that each group gets 0 or more objects (empty groups are allowed) = ${}^{n+r-1}C_{r-1}$.

Proof :

Let $x_1, x_2, x_3, \ldots, x_r$ be the number of objects given to groups 1, 2, 3, . . . , r respectively.

As total objects to be divided is n, we can take

Sum of the objects given to all groups = n

$\Rightarrow$ $x_1 + x_2 + x_3 + x_4 + \ldots + x_r = n$.

This equation is known as integral equation as all variables are integer.

As each group can get 0 or more, following are constraints on integer variables.

$0 \le x_1 \le n$; $0 \le x_2 \le n, \ldots,$ $0 \le x_r \le n$ *i.e.* $0 \le x_i \le n$ $i = 1, 2, 3, \ldots, r$.

We can observe that number of integral solutions of the above equation is equal to number of ways to divide n identical objects among r groups such that each gets 0 or more.